JOURNAL OF APPROXIMATION THEORY 34, 277–285 (1982)

Markov and Bernstein's Inequalities, and Compact and Strictly Singular Operators

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Communicated by E. W. Cheney

Received February 18, 1981

"Many theorems of approximation theory depend on the fact that a polynomial of degree n cannot change too rapidly; in other words, its derivative cannot be too large," Lorentz remarked in Ref. [11, p. 39]. Quantitative statements of this fact are given by two classical inequalities [11, pp. 39–41; 3, pp. 89–91].

Markov's inequality. If P_n is a polynomial of degree at most n, then

$$\|P_n'\| \leqslant n^2 \|P_n\|, \tag{1}$$

the norm being the supremum norm on $[-1, 1]$. Inequality (1) becomes an equality for the nth degree Chebyshev polynomial $\cos(n \arccos(x))$. (A nice discussion of the origin of (1) in a problem in chemistry considered by Mendeleev, the inventor of the periodic table, is given in [1]).

Bernstein's inequality. If T_n is a trigonometric polynomial of degree at most n, $(T_n(\theta) = \sum_1^n a_k \sin k\theta + \sum_0^n b_k \cos k\theta)$, then

$$\|T_n'\| \leqslant n \|T_n\|, \tag{2}$$

the norm being the supremum norm on the unit circle Γ. Inequality (2) becomes an equality for $\sin n\theta$ and $\cos n\theta$.

Both of these inequalities have the same form: A Banach space and a finite dimensional subspace M in the domain of the derivative D are given—for Markov's inequality M is the $n+1$-dimensional subspace of $C[-1, 1]$ of polynomials of degree less than or equal to n, for Bernstein's inequality M is the $2n+1$-dimensional subspace of $C(\Gamma)$ of trigonometric polynomials of degree less than or equal to n—and the conclusion states the value, respectively n^2 and n, of the norm $\|D|_M\|$ of D restricted to M.

From this point of view it is natural to ask to what extent the constants n^2 and n in inequalities (1) and (2) depend on the classes of functions,

0021-9045/82/030277–09\$02.00/0

polynomials or trigonometric polynomials. In particular, what improvement is possible, i.e., what is the best possible constant

$$d_n' = \inf_M\{\|D|_M\|: M \text{ an } n\text{-dimensional subspace in the domain of } D\}? \qquad (*)$$

This question turns out to have significant contact with several areas of approximation theory and operator theory.

These numbers d_n' also arise when one considers the Kolmogorov n-width d_n, defined for a convex symmetric subset A of a Banach space X by

$$d_n(A; X) = d_n(A) = \inf_N \sup_a \{d(a, N): a \text{ in } A, \; N \text{ an } n\text{-dimensional subspace of } X\}$$

[11, p. 132]. This measure of how well elements of A can be approximated by an n-dimensional subspace was introduced by Kolmogorov, who "insists that the determination of the *exact* value of $d_n(A)$ is important because it may lead to the discovery of extremal subspaces and therefore to new and better methods of approximation" [11, p. 133].

One powerful method of obtaining a lower bound for $d_n(A)$, due to Tikhomirov [16], proceeds as follows. If L is a subspace of dimension $n + 1$, then $d_n(A) \geqslant \inf_N \sup_x \{d(x, N): x \text{ in } A \cap L\}$. By a basic lemma [11, p. 137; 16, p. 78; 2], since $\dim N < \dim L$, there is an x in L with $\|x\| = d(x, N)$. Therefore, letting S_L denote the unit sphere in L, and setting

$$b_{n+1}(A) = \sup_L \sup_\lambda \{\lambda \geqslant 0: \lambda S_L \subseteq A, \dim L = n + 1\} \qquad (3)$$

we obtain

$$d_n(A) \geqslant b_{n+1}(A). \qquad (4)$$

(Often what we call b_{n+1} in (3) is called b_n, so that (4) becomes $d_n \geqslant b_n$.)

Tikhomirov calls $b_n(A)$ the Bernstein diameter, because these numbers are "very often encountered in the study of Bernstein-type inequalities" [16, p. 99]. In fact, let $B = \{f \text{ in } X: f' \text{ is in } X, \text{ and } \|f'\| \leqslant 1\}$, for X either $C[-1, 1]$ or $C(\Gamma)$. Then the number d_n' of $(*)$ is given by $d_n' = 1/b_n(B; X)$. We now compute this number for $C[-1, 1]$.

1. THEOREM. *Let M be a subspace of $C[-1, 1]$ consisting of continuously differentiable functions. If the dimension of M is at least $n + 1$, then there is a non-zero f in M satisfying*

$$\|f'\| \geqslant n\,\|f\|. \qquad (5)$$

The constant n in (5) is the best possible.

Proof. Set $r_1 = -1 + 1/n$ and $r_j = r_{j-1} + 2/n$ for $j = 2, 3, \ldots, n$. The map of M into E^n given by $f \to (f(r_1), \ldots, f(r_n))$ cannot be one-to-one, and thus there is a nonzero f in M with $f(r_j) = 0$ for all j. This function f attains its norm on $[-1, 1]$ at a point x. For the correct choice j of index, $|x - r_j| \leqslant 1/n$. Using the mean value theorem, $n\,\|f\| \leqslant |f(x) - f(r_j)|/|x - r_j| = |f'(\xi)| \leqslant \|f'\|$.

Set $t_j = -1 + 2j/n$, $j = 0, 1, \ldots, n$. Let $\varepsilon > 0$ be given and consider the function g_1 defined by

$$\begin{aligned} g_1(x) &= 0, && x \leqslant t_0 \\ &= a, && x = t_0 + \varepsilon \text{ and } x = t_1 - \varepsilon \\ &= -a, && x = t_1 + \varepsilon \text{ and } x = t_2 - \varepsilon \\ &= 0, && x \geqslant t_2, \text{ and linear otherwise,} \end{aligned}$$

where $a = (2/n - \varepsilon)^{-1}$, and we suppose that $\varepsilon < 2/n$.

Set $f_1(x) = \int_{-1}^{x} g(t)\,dt$. The function f_1 is non-negative, zero outside $[t_0, t_2]$, has a maximum of 1 at t_1, and for $t_1 \leqslant x \leqslant t_2$, $f_1(x) = 1 - f_1(x - 2/n)$. For $j = 0, 1, \ldots, n$, define $f_j(x) = f_1(x + 2(j-1)/n)$. It is important to note that for $j = 0$ and for $j = n$ one-half of the function f_j is shifted off the interval $[-1, 1]$.

Take $M = \text{sp}(f_0, f_1, \ldots, f_n)$. For $\sum c_j f_j = f$ in M, we compute the norm: If $t_j \leqslant x \leqslant t_{j+1}$, then $f(x) = c_j f_j(x) + c_{j+1} f_{j+1}(x) = c_j f_j(x) + c_{j+1} f_j(x - 2/n) = c_j f_j(x) + c_{j+1}(1 - f_j(x))$. (In passing, note that $1 = \sum f_j$ belongs to M.) As the range of $f_j(x)$, $t_j \leqslant x \leqslant t_{j+1}$ is $[0, 1]$, $\max\{|f(x)| : t_j \leqslant x \leqslant t_{j+1}\} = \max(|c_j|, |c_{j+1}|)$. Hence $\|\sum c_j f_j\| = \max\{|c_j| : 0 \leqslant j \leqslant n\}$ (consequently, $\dim M = n + 1$).

Now we compute the norm of f'. For $t_j \leqslant x \leqslant t_{j+1}$, $|f'(x)| = |c_j g_j(x) + c_{j+1} g_{j+1}(x)| = |c_j g_j(x) - c_{j+1} g_j(x)|$. Thus $\max\{|f'(x)| : t_j \leqslant x \leqslant t_{j+1}\} = a\,|c_j - c_{j+1}|$. Hence $\|\sum c_j f_j'\| = a \max\{|c_{j+1} - c_j| : 0 \leqslant j \leqslant n - 1\} \leqslant 2a\,\|f\| = n(1 - n\varepsilon/2)^{-1}\,\|f\|$, and the fact that the constant n in (3) is best possible follows. Q.E.D.

To restate Theorem 1: In $C[-1, 1]$, $d_{n+1} = n$. Note the improvement over the constant n^2 of Markov's theorem.

Theorem 1 is close to Tikhomirov's calculation of d_n for $W_1 = \{f$ in $C[-\pi, \pi] : f'$ is in $L^\infty[-\pi, \pi]$, and $\operatorname{ess\,sup} |f'| \leqslant 1\}$ [16, p. 81]; indeed, inequality (5) follows from his result. That n is the best constant requires additional argument, like that given above, where the idea behind the construction of the subspace which shows that (5) is sharp is to construct a subspace $M = \text{sp}(h_0, \ldots, h_n)$, where the continuous functions h_j are differentiable except for a finite number of points, e.g., h_1 an isoceles triangle of height 1 and base $[t_0, t_2]$, and where (5) holds except at the points of nondifferentiability. Then by rounding the corners on the h_j you can get (5) to hold to within ε and the h_j will be as smooth as you like, in particular they will be continuously differentiable and so will be in the domain of D.

P. Wojtaszczyk has shown, in a private communication, that there is no extremal subspace in $C[-1, 1]$ with best constant n in (5). I believe that Tikhomirov considers W_1, rather than the more natural $\{f \text{ in } C[-\pi, \pi] : f' \text{ is in } C[-\pi, \pi] \text{ and } \|f'\| \leqslant 1\}$, just to circumvent this fact and obtain extremal subspaces [16, p. 82], but at the price of changing the range of the derivative.

2. THEOREM. *Let M be a subspace of continuously differentiable functions on the unit circle Γ.*

(a) *If* $\dim M = 2k$, *then there is a non-zero function f in M with* $\|f'\| \geqslant 2k/\pi \, \|f\|$. *This constant is best possible.*

(b) *If* $\dim M = 2k + 1$, *there is a non-zero f in M with* $\|f'\| \geqslant 2k/\pi \, \|f\|$. *Further, given $\varepsilon > 0$ there is a subspace M of dimension $2k + 1$ with* $\|f'\| \leqslant (2k + 1 + \varepsilon)/\pi \, \|f\|$ *for all f in M.*

Proof. First suppose that $\dim M = 2k$, $M = \operatorname{sp}(f_1, f_2, \ldots, f_{2k})$. For $0 \leqslant \theta \leqslant 2\pi$ define $Z(\theta)$ to be the determinant of the $2k \times 2k$ matrix $[f_c(\exp(i(\theta + r\pi/k)))]$, r the row index, c the column index. The determinant $Z(\theta)$ can be obtained from $Z(\theta + \pi/k)$ by $2k - 1$ row interchanges, so $Z(\theta) = -Z(\theta + \pi/k)$; it follows that $Z(\theta)$ has a zero θ_0 in $[0, \pi/k)$. Let $a_1, \ldots, a_{2k}$ be a non-trivial solution to the system of homogeneous equations which has $Z(\theta_0)$ as its coefficient determinant. Then $f = a_1 f_1 + \cdots + a_{2k} f_{2k}$ is a nonzero element of M with $2k$ equally spaced zeros $r_m = \exp(i(\theta_0 + m\pi/k))$ $m = 1, 2, \ldots, 2k$. If f attains its norm at x on Γ, then, identifying Γ with $[0, 2\pi]$, $|x - r_j| \leqslant \pi/2k$ for some index j. By the mean value theorem, $2k/\pi \, \|f\| \leqslant |f(x) - f(r_j)|/|x - r_j| = |f'(\xi)| \leqslant \|f'\|$.

If $\dim M = 2k + 1$, inequality (b) follows from inequality (a).

The example used to establish the given upper bounds on d'_n comes from the example of Theorem 1 by changing $[-1, 1]$ to $[0, 2\pi]$ and then identifying $[0, 2\pi]$ with Γ. Note that in this process the two functions f_0 and f_n in $C[-1, 1]$ become one function in $C(\Gamma)$, lowering the dimension of the subspace M by one. Q.E.D.

To restate Theorem 2: In $C(\Gamma)$, $d'_{2k} = 2k/\pi$ and $2k/\pi \leqslant d'_{2k+1} \leqslant (2k + 1)/\pi$. Compare these values with Bernstein's inequality. Compare Theorem 2 with Tikhomirovs calculation of d_n for $\tilde{W}_1 = \{f \text{ in } C(\Gamma) \colon f' \in L^\infty(\Gamma) \text{ and } \operatorname{ess\,sup} |f'| \leqslant 1\}$ [16, p. 94].

The subspace $M = \operatorname{sp}(1, \sin\theta, \cos\theta, \ldots, \cos k\theta\}$ of dimension $2k + 1$ satisfies the Haar condition [3, p. 94] and so each non-zero f in M can have at most $2k$ zeros; therefore the general method of proof of (a) cannot be extended to get $d'_{2k+1} = (2k+1)/\pi$. What is the value of d'_{2k+1}?

The value of d'_{2k} cannot be attained by a subspace M of dimension $2k$. For if it were attained for a subspace M, then we may take $f(x_0) = 1 = \|f\|$ in the function f constructed in the proof of Theorem 2. Note that on $r_j \leqslant x_0 \leqslant r_{j+1}$ f goes from 0 to 1 and back to zero with $\|f'\| \leqslant 2k/\pi$. So f must be linear from r_j to x_0 and from x_0 to r_{j+1} and therefore is not differentiable at x_0.

Look at the most basic property of the sequence d'_n; in the Banach spaces $C[-1, 1]$ and $C(\Gamma)$ the derivative D is uniformly unbounded in the sense that d'_n tends to infinity. Why? That is, what is the essential property of the operator D which corresponds to this behavior of d'_n? A good answer would tell us, without calculation, whether d'_n tends to infinity in other Banach spaces. But deeper than that, a good answer would identify a class of operators, and this class would probably be interesting and useful as it would be defined in terms of a natural property, of the basic operator D, which is suggested by the important Markov and Bernstein's inequalities.

The first step towards answering this question is to note that differentiation is the inverse of integration. And integration is easier to work with than D, as it is a bounded operator with domain the whole space, while D is closed and densely defined. Since the indefinite integral is only defined to within a constant, one must be more precise, choosing a constant a in the base space and considering T_a defined by $T_a f(x) = \int_a^x f(t)\,dt$. The inverse of T_a is D_a, which is D restricted to those functions f in its domain which satisfy $f(a) = 0$, i.e., D_a is D together with an initial condition.

Define

$$d'_n(D_a) = \inf_M \{\|D_a|_M\| : M \text{ an } n\text{-dimensional subspace in the domain of } D_a\}. \tag{6}$$

3. Lemma. *The inequalities $d'_n \leqslant d'_n(D_a) \leqslant d'_{n+1} \leqslant d'_{n+1}(D_a)$ hold.*

Proof. First, $d'_n = \inf\{\|D|_M\| : \dim M = n,\ M$ in the domain of $D\} \leqslant \inf\{\|D|_M\| : \dim M = n,\ M$ in the domain of $D_a\}$, since the domain of D_a is contained in the domain of D.

Second, let N_a be those f in the domain of D satisfying $f(a) = 0$. If $\dim M = n+1$, M in the domain of D, then

$$\|D|_M\| \geqslant \|D|_{M\cap N_a}\|. \tag{7}$$

Since $\dim(M \cap N_a)$ is either n or $n+1$, the RHS of (7) is bounded below by $d'_n(D_a)$ or $d'_{n+1}(D_a)$, and so by $d'_n(D_a)$. Hence $d'_{n+1} \geqslant d'_n(D_a)$. Q.E.D.

So now we will know why $d'_n \to \infty$ if we know why $d'_n(D_a) \to \infty$.

The next step is to generalize $(*)$ in the obvious way. Let X and Y be two Banach spaces and $T: X \to Y$ a bounded linear operator. For the closed linear operator $S = T^{-1}$, with domain the range of T, define

$$d_n'(S) = \inf\{\|S|_M\| : \dim M = n, M \text{ in the domain of } S\}. \tag{8}$$

This could, of course, also be written in terms of Tikhomirov's Bernstein diameters. We want to rewrite (8) in terms of T.

The injection modulus for a bounded linear operator $T: X \to Y$ is defined by [13, p. 26]

$$j(T) = \sup\{\lambda : \|Tx\| \geqslant \lambda \|x\| \text{ for all } x\}.$$

For T one-to-one, $j(T) = 1/\|T^{-1}\|$.

4. Lemma. *Let $T: X \to Y$ be one-to-one, $S = T^{-1}$, and $d_n'(S)$ be as above. Then*

$$1/d_n'(S) = \sup\{j(T|_M) : \dim M = n\}. \tag{9}$$

Proof. $d_n'(s) = \inf[\|T^{-1}|_M\| : \dim M = n] = \inf_M[\sup\{\|T^{-1}x\|/\|x\| : 0 \neq x$ in $M\} : \dim M = n] = \inf_M[\sup\{\|y\|/\|Ty\| : 0 \neq y \text{ in } T^{-1}(M)\} : \dim M = n] = \inf_N[1/j(T|_N) : \dim N = n]$, and the lemma follows. Q.E.D.

If we now focus our attention on T, rather than on $S = T^{-1}$, then the requirement that T be one-to-one is superfluous, and we can define, for any bounded linear operator $T: X \to Y$, the numbers

$$u_n(T) = \sup[j(T|_M) : \dim M = n]. \tag{10}$$

This brings us into contact with Pietsch's work [14, p. 207], where the numbers $u_n(T)$ are introduced and there called the Bernstein numbers of T. In terms of this formulation, our original question of why d_n' tends to infinity becomes: Why does $u_n(T_a)$ tend to zero? To understand this question in context, recall that on a separable Hilbert space H the operator ideals can be described in terms of s-numbers $s_n(T)$ of an operator T [4, p. 1089]. The operator T is compact iff $s_n(T) \to 0$. For compact T, $s_n(T)$ is the nth eigenvalue λ_n of the positive operator $(T^*T)^{1/2}$. The von Neumann–Schatten ideals C_p, consisting of all T with $\sum s_n(T)^p < \infty$, $0 < p < \infty$, generalize the classical trace class ($p = 1$) and Hilbert–Schmidt ($p = 2$) ideals. Pietsch's way of extending this ideal theory to operators on a Banach space is to replace the sequence $\{\lambda_n\}$ of eigenvalues of $(T^*T)^{1/2}$ by a sequence $\{s_n(T)\}$ having certain properties.

From the general results of [14], it follows that for T on a Hilbert space, $u_n(T) \to 0$ iff T is compact. (And an easy calculation for compact T, using the spectral theorem for $(T^*T)^{1/2}$, shows that $u_n(T) = \lambda_n$, which also follows from [14, Theorem 2.1, p. 203].) But the geometry of a Banach space, and the resulting operator theory, is considerably more complicated, and there are many distinct choices for $\{s_n(T)\}$, all agreeing with $\{\lambda_n\}$ on H [14, Theorem 2.1, p. 203]. However, one result which is true in general is that T compact implies that $u_n(T) \to 0$; see below.

In [8] Kato introduced strictly singular operators, a generalization of compact operators. A good discussion of their use in perturbation theory is [6]. One definition is that T is strictly singular if it does not have a bounded inverse on any infinite dimensional subspace [6, p. 76]; a definition which makes it seem plausible that $\mathscr{U} = \{T: X \to Y: u_n(T) \to 0\}$ is the class of strictly singular operators.

5. THEOREM. *For T compact, $u_n(T) \to 0$. If $u_n(T) \to 0$, then T is strictly singular.*

Proof. The quickest proof is to note that the Gelfand numbers $g_n(T) \to 0$ for T compact [14, Theorem 9.3, p. 220] and $g_n(T) \geqslant u_n(T)$ [14, Theorems 4.4, 4.5, p. 207]. The following alternate proof indicates a connection between the numbers $u_n(T)$ and Kolmogorov's notion of the capacity of a compact set [11, p. 150]. Since T is compact, the image TS_X, of the unit ball S_X of X, is compact. By the definition of the capacity $C_\epsilon = \log N_\epsilon$, for each $\varepsilon > 0$ there is an ε-net $\{Tx_i: \|x_i\| \leqslant 1,\ 1 \leqslant i \leqslant N_\epsilon\}$ for TS_X. There are linear functionals x_i^* with $x_i^*(Tx_i) = \|Tx_i\|$, $\|x_i^*\| = 1$, for $1 \leqslant i \leqslant N_\epsilon$. If M is any subspace of dimension greater than N_ϵ, there is a norm one x in M with $x_i^*(Tx) = 0$ for $1 \leqslant i \leqslant N_\epsilon$. For some index j, $\|Tx - Tx_j\| \leqslant \varepsilon$. Hence $\|Tx\| \leqslant \varepsilon + \|Tx_j\| = x_j^*(Tx_j) + \varepsilon = |x_j^*(Tx_j) - x_j^*(Tx)| + \varepsilon \leqslant 2\varepsilon$. Thus $u_n(T) \leqslant 2\varepsilon$ for $n > N\varepsilon$.

Suppose that T is not strictly singular, and so has a bounded inverse on an infinite dimensional subspace N. For M an n-dimensional subspace of N we have $u_n(T) \geqslant j(T|_M) \geqslant 1/\|(T|_N)^{-1}\|$, and $u_n(T)$ does not tend to zero. Q.E.D.

In answer to the obvious questions raised by Theorem 5:

6. EXAMPLE. (a) *There is an operator I_1 which is not compact and yet has $u_n(I_1) \to 0$.*

(b) There is a strictly singular operator I_2 with $u_n(I_2) \not\to 0$.

Elements in the sequences spaces l^1 and c_0 will be denoted by $x = (x(1), x(2), \cdots)$. The injection $I_1: l^1 \to c_0$ is not compact. Assume that there is a number $b > 0$ with $u_n(I_1) > b$ for all n. Let m be given. By hypothesis there is a subspace M in l^1 of dimension greater than $[2^2/b] + 1 + [2^3/b] + 1 + \cdots + [2^m/b] + 1$ with $\|I_1 x\| = \|x\|_\infty > b\|x\|_1$ for all x in M. There is x_1 in M with $\|x_1\|_1 = 1$, and $\|x_1\|_\infty > b$ with $|x_1(n_1)| > b$.

Let $N_1 = \{j: |x(j)| > b/2^2\}$. Since $\sum |x_1(j)| = 1$, N_1 contains less than $[2^2/b] + 1$ integers. Because the dimension of M is greater than $[2^2/b] + 1$ there is an x_2 in M, $\|x_2\|_1 = 1$, with $x_2(j) = 0$ for j in N_1. For this x_2, $\|x_2\|_\infty > b$, and there is an n_2 with $|x_2(n_2)| > b$. Let $N_2 = \{j: |x_2(j)| > b/2^3\}$. Since $\sum |x_2(j)| = 1$, N_2 contains less than $[2^3/b] + 1$ integers. Since the dimension of M is greater than $[2^2/b] + 1 + [2^3/b] + 1$, there is an x_3 in M, $\|x_3\|_1 = 1$, with $x_3(j) = 0$ for j in $N_1 \cup N_2$. For this x_3, $\|x_3\|_\infty > b$ and $|x_3(n_3)| > b$. Let $N_3 = \{j: |x_3(j)| > b/2^4\}$, a set of fewer than $[2^4/b] + 1$ integers. Continue, obtaining $x_1, x_2, \ldots, x_m$. Consider the element $\sum_1^m x_k$ in M. For j in N_i, $|\sum x_k(j)| \leqslant |x_i(j)| + \sum_{k \neq i} |x_k(j)| \leqslant 1 + b/2$, whereas if j does not belong to $\bigcup N_i$, then $|\sum x_k(j)| \leqslant b/2$. Thus $\|\sum x_k\|_\infty \leqslant 1 + b/2$. On the other hand,

$$\left\| \sum x_k \right\|_1 \geqslant \sum_j \left| \sum_k x_k(n_j) \right| \geqslant \sum_j \left(b - \sum_{k \neq j} |x_k(j)| \right) \geqslant \sum_1^m b/2 = mb/2.$$

So we have $1 + b/2 \geqslant \|\sum x_k\|_\infty \geqslant b \|\sum x_k\|_1 \geqslant mb^2/2$, which is a contradiction for sufficiently large m.

Let E^n denote n-dimensional Euclidean space. Let $X = (\sum E^n)_{l^1}$: an element x in X is then a sequence $\{x_n\}$, x_n in E^n, with $\|x\| = \sum \|x_n\| < \infty$. Let $Y = (\sum E^n)_{c_0}$: an element y in Y is then a sequence $\{y_n\}$, with y_n in E^n, $\|y_n\| \to 0$, and $\|y\| = \sup \|y_n\|$. Let I_2 be the injection of X into Y. The quickest way to see that I_2 is strictly singular is to note that X is isomorphic to a subspace of l^1 and Y is isomorphic to a subspace of c_0 (15, pp. 304–306], and I_2 is then strictly singular as no infinite dimensional subspace of l^1 can be isomorphic to a subspace of c_0; see, e.g., [10]. Looking at the action of I_2 on E^n we see that $u_n(I_2) \geqslant 1$ for each n. Q.E.D.

Whenever the operator $T_a f(x) = \int_a^x f(t)\,dt$ is compact, we see that $d_n' \to \infty$. This is true in the spaces $C[-1, 1]$ and $C(\Gamma)$ which we considered, as well as in spaces we have not considered, e.g., L^p[5, 12, 16, 17]. This constitutes one reasonable explanation of why d_n' tends to infinity, but an adequate amount of mystery still remains. It is not known whether the set of operators $\mathscr{U} = \{T: X \to Y: u_n(T) \to 0\}$, trapped between the ideals of compact and strictly singular operators, is itself an ideal [14, p. 222]. If not always an ideal, when is it an ideal? When is $\mathscr{U}$ the compact operators or the strictly singular operators? Can $\mathscr{U}$ be characterized in some interesting way?

Pietsch [14, p. 220] shows that an operator T is compact iff $s_n(T) \to 0$ for s_n either the Kolmogorov numbers or the Gelfand numbers. Example 6 suggests the problem of finding a sequence k_n of s-numbers with the property that T is strictly singular iff $k_n(T) \to 0$.

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